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Quantum dynamics of a solvable nonlinear chiral model

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Abstract. The quantum mechanical analogue of the classical nonlinear system with the Lagrangian

$$L = \frac{1}{2}\left(\dot{\boldsymbol{q}}^2 + \frac{\lambda(\boldsymbol{q}\cdot\dot{\boldsymbol{q}})^2}{1-\lambda\boldsymbol{q}^2} - \frac{k_0\boldsymbol{q}^2}{1-\lambda\boldsymbol{q}^2}\right)$$

is shown to be exactly solvable and its energy levels and eigenfunctions are obtained completely. The symmetric version ($k_0 = 0$) of this model (obeying nonlinear transformation laws with the coordinates q_i, $i = 1, 2, 3$ parametrizing a projective three-sphere of radius $1/\sqrt{\lambda}$ when $\lambda > 0$ and $|q| \leqslant \lambda^{-1/2}$) is the SU(2)⊗SU(2) chiral invariant Lagrangian in the Gasiorowicz–Geffen coordinates. The radial part of the classical equation of motion (in both the symmetric and non-symmetric cases) admits simple harmonic bounded solutions and the bound state energies of the quantized system show a linear dependence on the coupling parameter λ. It is shown that the Bohr–Sommerfeld quantization procedure reproduces the form of the correct bound state energy levels while a perturbation theoretic treatment gives the exact energy expressions. Arguing along the lines of Velo and Wess that the Hamiltonian be invariant under the action of the underlying internal symmetry group, the ordering problem that arises in the quantum mechanical case is overcome. The results are in agreement with those of Lin, Lin and Sugano.

1. Introduction

Evaluation of the S matrix in perturbation theory for phenomenological Lagrangian models describing pion interactions presents formidable problems: the theory is non-renormalizable (Delbourgo *et al* 1969, Keck and Taylor 1973); equivalent interactions possess highly singular terms (proportional to $\delta^4(0)$, see eg Gerstein *et al* 1971) and there are new ordering problems that have to be taken care of (Charap 1973a). An important feature of these chiral models is the nonlinear transformation law for the field operators under the action of the internal symmetry group (Gasiorowicz and Geffen 1969). Simplified versions of these models in zero-space dimensions which retain the above-mentioned transformation properties are of interest in this connection. Besides having an intrinsic value, these models enable one to understand certain basic problems more clearly, eg the validity of the perturbation theory, the structure of matrix elements of field operators, the ordering problems arising from the derivative interaction terms, etc. Some work along these lines has already appeared: Velo and Wess (1971) have investigated a model in the Weinberg coordinates—the symmetric model by group theoretical methods and a non-symmetric one by perturbative methods, since it was not amenable to exact solution; Charap (1973b) has studied a model with tangential parametrization

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in the massless case to obtain the energy eigenvalues and eigenfunctions; Andreev (1973) has considered a classical σ model. Recently we have solved (Mathews and Lakshmanan 1975) exactly the quantum mechanical problem of the isoscalar version of the SU(2)$\otimes$ SU(2) chiral model in the Gasiorowicz–Geffen coordinates (Delbourgo *et al* 1969) with the Lagrangian

$$L = \frac{1}{2}\left(\dot{\boldsymbol{q}}^2 + \frac{\lambda(\boldsymbol{q}\cdot\dot{\boldsymbol{q}})^2}{1-\lambda\boldsymbol{q}^2} - \frac{k_0\boldsymbol{q}^2}{1-\lambda\boldsymbol{q}^2}\right). \tag{1}$$

Our main objective here is to present the solution of the quantum mechanical problem of this isotriplet model (1) with all the three degrees of freedom included and the symmetry breaking term ($k_0 \neq 0$) also added.

The important feature of this model is that it is *exactly* solvable, even in the non-symmetric case. The classical bound state solutions are simple harmonic; the quantum mechanical bound state energy expressions show a simple linear dependence on the coupling parameter and perturbation theory is valid and the Bohr–Sommerfeld quantization reproduces the form of the exact bound state energy level expressions. The plan of the paper is as follows. We solve in § 2 the classical equation of motion of the system (1) and show that the radial oscillations are simple harmonic. We then use a modified Bohr–Sommerfeld quantization rule to obtain the approximate bound state energy levels. We obtain in § 3 the unique quantum Hamiltonian for the system (1) by imposing the invariance of the Hamiltonian under the action of the chiral SU(2)$\otimes$SU(2) group for the massless ($k_0 = 0$) case. This Hamiltonian contains even in this massless case a purely coordinate-dependent non-polynomial term. In the appendix we prove that this extra term is exactly equivalent to the one that is obtained from the general theory of Lin *et al* (1970), where the consistency between the Lagrangian and Hamiltonian formalisms is the main criterion. In § 4, the Schrödinger equation is solved for both the bound states and continuum states. The solutions are shown to have the correct $\lambda \to 0$ limit. Section 5 gives a brief discussion of the perturbation theoretic result in comparison with the exact one.

2. Classical solutions and semiclassical quantization

The Euler–Lagrange equation of motion of the system (1) is

$$\ddot{q}_i + \left(\frac{\lambda(\boldsymbol{q}\cdot\ddot{\boldsymbol{q}})}{1-\lambda\boldsymbol{q}^2} + \frac{\lambda\dot{\boldsymbol{q}}^2}{1-\lambda\boldsymbol{q}^2} + \frac{\lambda^2(\boldsymbol{q}\cdot\dot{\boldsymbol{q}})^2}{(1-\lambda\boldsymbol{q}^2)^2} + \frac{k_0}{(1-\lambda\boldsymbol{q}^2)^2}\right)q_i = 0. \tag{2}$$

With the introduction of the polar coordinates

$$\begin{aligned} q_1 &= q\sin\theta\cos\varphi \\ q_2 &= q\sin\theta\sin\varphi \\ q_3 &= q\cos\theta \end{aligned} \tag{3}$$

equation (2) separates out into the following:

$$q^2\sin^2\theta\dot{\varphi} = C_1 = \text{constant} \tag{4}$$

$$q^4\dot{\theta}^2 + C_1^2/\sin^2\theta = C_2^2 = \text{constant} \tag{5}$$

and

$$\ddot{q}+\frac{\lambda q(\dot{q})^2}{1-\lambda q^2}+\frac{k_0 q}{1-\lambda q^2} = \frac{C_2^2(1-\lambda q^2)}{q^3}. \tag{6}$$

Integrating equation (6) once, we have

$$\frac{\dot{q}^2}{1-\lambda q^2}+\frac{k_0}{\lambda}\frac{1}{1-\lambda q^2}+\frac{C_2^2}{q^2} = C_3 = \text{constant}. \tag{7}$$

The case $C_2 = 0$ corresponds to the isoscalar case considered (Mathews and Lakshmanan 1974) by us previously. By proceeding along the lines of Mathews and Lakshmanan (1974) we can easily show that the periodic solutions to the radial equation (7) are given by

$$q(t) = A[1-\beta\sin^2(\omega t+\zeta)]^{1/2} \tag{8a}$$

where

$$\omega^2 = \frac{k_0}{1-\lambda A^2}+\frac{\lambda C_2^2}{A^2} \tag{8b}$$

and

$$\beta = 1-\frac{1}{\lambda A^2}\left(1-\lambda A^2-\frac{k_0-\lambda^2 C_2^2}{\omega^2}\right) \tag{8c}$$

with

$$A^2 = c+\left(c^2-\frac{C_2^2}{\lambda C_3}\right)^{1/2} \tag{9a}$$

where

$$c = \frac{1}{2\lambda}\left(1-\frac{k_0-\lambda^2 C_2^2}{\lambda C_3}\right). \tag{9b}$$

We can easily see that (when $\lambda > 0$) the range of these periodic solutions is such that $0 \leqslant A \leqslant \lambda^{-1/2}$. If we consider also the values of $|q(t)|$ exceeding this range ($\lambda^{-1/2}$) (which is not included in the underlying SU(2) ⊗ SU(2) manifold) one obtains aperiodic motions.

The canonically conjugate momenta for the Lagrangian (1) are

$$\boldsymbol{p} = \dot{\boldsymbol{q}}+\frac{\lambda(\boldsymbol{q}\cdot\dot{\boldsymbol{q}})}{1-\lambda\boldsymbol{q}^2}\boldsymbol{q} \tag{10}$$

so that the classical Hamiltonian becomes

$$H = \frac{1}{2}\left(\dot{\boldsymbol{q}}^2+\frac{\lambda(\boldsymbol{q}\cdot\dot{\boldsymbol{q}})^2}{1-\lambda\boldsymbol{q}^2}+\frac{k_0\boldsymbol{q}^2}{1-\lambda\boldsymbol{q}^2}\right)$$
$$= \frac{1}{2}\left(\boldsymbol{p}^2-\lambda(\boldsymbol{p}\cdot\boldsymbol{q})^2+\frac{k_0\boldsymbol{q}^2}{1-\lambda\boldsymbol{q}^2}\right). \tag{11}$$

To find the approximate bound state energies by semiclassical methods we proceed as below. Using the expressions (4) and (5) in (11) we have

$$H = \frac{1}{2}\left(\frac{\dot{q}^2 + k_0 q^2}{1-\lambda q^2} + \frac{C_2^2}{q^2}\right). \tag{12}$$

On substitution of the periodic solution (8) in (12) we obtain the classical energy expression to the system (1) as

$$E = \frac{1}{2}\left(\frac{k_0 A^2}{1-\lambda A^2} + \frac{C_2^2}{A^2}\right) = \frac{1}{2}\left(\frac{\omega^2}{\lambda} - \frac{k_0}{\lambda}\right). \tag{13}$$

Then assuming

$$C_1 = m\hbar, \qquad C_2^2 = (l+\tfrac{1}{2})^2\hbar^2 \tag{14}$$

we apply the Bohr–Sommerfeld quantization rule for the radial part. We then obtain from the rule†

$$\oint p_r \, \mathrm{d}r = (n_r + \tfrac{1}{2})h \qquad (n_r = 0, 1, 2, \ldots) \tag{15}$$

that

$$\oint \frac{\dot{q}}{1-\lambda q^2}\frac{\mathrm{d}q}{\mathrm{d}\eta}\,\mathrm{d}\eta = (n_r + \tfrac{1}{2})h \qquad (\eta = \omega t + \zeta). \tag{16a}$$

Or we have

$$\omega A^2 \beta^2 \int_0^{\pi} \frac{\cos^2\eta \sin^2\eta \, \mathrm{d}\eta}{(1+\beta\sin^2\eta)(1-\lambda A^2 + \lambda\beta A^2 \sin^2\eta)} = (n_r + \tfrac{1}{2})h. \tag{16b}$$

Then after some simplification we obtain

$$\oint p_r \, \mathrm{d}r = \frac{\omega}{\lambda}\pi - \omega A^2(1-\beta)I_1 - \frac{\omega}{\lambda}\surd(1-\lambda A^2)[1-\lambda A^2(1-\beta)]I_2 \tag{17}$$

where

$$I_1 = \int_0^{\pi} \frac{\mathrm{d}\eta}{1-\beta\sin^2\eta} = \frac{1}{\surd(1-\beta^2)}\pi \tag{18a}$$

and

$$I_2 = \int_0^{\pi} \frac{\mathrm{d}\eta}{1-\lambda A^2(1-\beta\sin^2\eta)} = \frac{1}{\surd(1-\lambda A^2)}\,\frac{1}{\surd[1-\lambda A^2(1-\beta)]}\pi. \tag{18b}$$

Then equation (17) becomes

$$-\frac{\surd k_0}{\lambda} + \frac{\omega}{\lambda} - C_2 = (2n_r + 1)\hbar. \tag{19}$$

So the approximate energy level expression from (13) and (19) becomes

$$E_{n_r,l} = (2n_r + l + \tfrac{3}{2})k^{1/2}\hbar + \tfrac{1}{2}\lambda(2n_r + l + \tfrac{3}{2})^2\hbar^2. \tag{20}$$

Later in § 4 we find that this expression closely approximates the exact one.

† The factor $\frac{1}{2}h$ on the right-hand side of (15) is added to obtain the correct $\lambda \to 0$ limit.

3. The quantum Hamiltonian

We now obtain a unique quantum Hamiltonian for the classical system (1) by requiring the invariance of the Hamiltonian in the symmetric case ($k_0 = 0$) under the action of the chiral SU(2) ⊗ SU(2) group, a procedure that was adopted by Velo and Wess (1971). We may easily show that the *nonlinear* transformation under which the Lagrangian (1) with $k_0 = 0$ is invariant (apart from the rotations in the isotopic space) is

$$\boldsymbol{q} \rightarrow \boldsymbol{q}' = \boldsymbol{q} + \delta\boldsymbol{q} \tag{21}$$

where

$$\delta\boldsymbol{q} = (1 - \lambda\boldsymbol{q}^2)^{1/2}\boldsymbol{\alpha}. \tag{22}$$

Here $\boldsymbol{\alpha}$ is a constant infinitesimal vector. This is obtained by taking the most general form of $\delta\boldsymbol{q}$ as

$$\delta\boldsymbol{q} = f(q^2)\boldsymbol{\alpha} + h(q^2)(\boldsymbol{\alpha} \cdot \boldsymbol{q})\boldsymbol{q} + g(q^2)(\boldsymbol{\alpha} \times \boldsymbol{q})$$

and solving for f, g and h. By Noether's theorem we then have that the generator which induces the nonlinear transformation (21) is

$$F_i = (1 - \lambda\boldsymbol{q}^2)^{1/2} p_i \tag{23}$$

where p_i are the canonically conjugate momenta. To go over to the quantum mechanical case we assume as usual that p_i and q_j are non-commuting operators in the Hilbert space obeying the commutation relations $[q_i, p_j] = i\hbar\,\delta_{ij}$, etc and that quantum mechanical operators such as F_i and H should be properly symmetrized to ensure Hermiticity.

The symmetrization of F_i poses no problem and is uniquely given by

$$F_i = \tfrac{1}{2}[(1 - \lambda\boldsymbol{q}^2)^{1/2} p_i + p_i(1 - \lambda\boldsymbol{q}^2)^{1/2}]. \tag{24}$$

On the other hand there are a number of possible ways by which the classical Hamiltonian $H = \frac{1}{2}[\boldsymbol{p}^2 - \lambda(\boldsymbol{p} \cdot \boldsymbol{q})^2]$ may be symmetrized; they differ from each other by coordinate-dependent terms. However the imposition of chiral invariance on the quantum system restricts the choice of the Hamiltonian to a unique one.

Now the angular momentum generators corresponding to rotations in isospace are given by

$$J_i = \epsilon_{ijk} q_j p_k. \tag{25}$$

Then after some algebra we may show that

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k \tag{26a}$$

$$[\lambda^{-1/2}F_i, \lambda^{-1/2}F_j] = \mathrm{i}\epsilon_{ijk} J_k \tag{26b}$$

and

$$[\lambda^{-1/2}F_i, J_j] = \mathrm{i}\epsilon_{ijk}\lambda^{-1/2}F_k. \tag{26c}$$

We also have the following quantum transformation laws:

$$[J_i, q_j] = \mathrm{i}\epsilon_{ijk} q_k \tag{27a}$$

and

$$[\lambda^{-1/2}F_i, q_j] = -\lambda^{-1/2}(1-\lambda q^2)^{1/2}\,\delta_{ij}. \tag{27b}$$

Thus† J_i are the triplet generators of the parity conserving isospin subgroup, while $\lambda^{-1/2}F_i$ are the pure chiral generators. We also note that the underlying group is either SO(3, 1) or SO(4) depending on whether $\lambda < 0$ or $\lambda > 0$ (see equation (26*b*)). The two Casimir operators in both the cases are given by

$$\begin{aligned} C_1 &= \frac{1}{\lambda}(F_iF_i + \lambda J_iJ_i) \\ &= \frac{2}{\lambda}\left(\frac{\boldsymbol{p}^2}{2} - \frac{1}{2}\lambda(\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{p}) + \frac{1}{8}\frac{\lambda^2\hbar^2\boldsymbol{q}^2}{1-\lambda\boldsymbol{q}^2}\right) + \frac{3}{2}\hbar^2 \end{aligned} \tag{28a}$$

and

$$C_2 = F_iJ_i = 0. \tag{28b}$$

In the evaluation of C_1 and C_2 we have made use of equations (24) and (25) and straightforward algebra. We now observe that the expression inside the large parentheses of equation (28*a*) is a possible symmetrization‡ of the corresponding classical Hamiltonian (11) with $k = 0$. Thus we identify our quantum Hamiltonian as

$$H = \frac{1}{2}\left(\boldsymbol{p}^2 - \lambda(\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{p}) + \frac{1}{4}\frac{\lambda^2\hbar^2\boldsymbol{q}^2}{1-\lambda\boldsymbol{q}^2}\right). \tag{29}$$

We note that this quantum Hamiltonian contains a non-polynomial coordinate-dependent term even in the absence of symmetry breaking terms. It is interesting to note at this point that the Hamiltonian expression (29) agrees exactly with the one that is obtained from the general theory of Lin *et al* (1970) for quantum mechanical velocity-dependent interactions where the consistency between the Lagrangian and Hamiltonian formalisms is insisted upon. The proof is given in the appendix. In this later theory the procedure is to find an equivalent velocity-independent Lagrangian where both the Lagrangian and Hamiltonian formalisms are equivalent.

4. Solutions of the quantum system

We assume the appropriate quantum Hamiltonian of our system (1) including the mass term ($k \neq 0$) in lieu of our discussion in § 3 to be

$$H = \tfrac{1}{2}\left(\boldsymbol{p}^2 - \lambda(\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{p}) + \frac{k\boldsymbol{q}^2}{1-\lambda\boldsymbol{q}^2}\right) \tag{30}$$

† In the classical case, one replaces the commutators by Poisson brackets.

‡ The classical Hamiltonian $H = \frac{1}{2}[\boldsymbol{p}^2 - \lambda(\boldsymbol{p}\cdot\boldsymbol{q})^2] = \frac{1}{2}\{\boldsymbol{p}^2(1-\lambda\boldsymbol{q}^2) - \lambda[-\boldsymbol{p}^2\boldsymbol{q}^2 + (\boldsymbol{p}\cdot\boldsymbol{q})^2]\}$ may be symmetrized as

$$\begin{aligned} H = \tfrac{1}{2}\{\tfrac{1}{4}[&p_i(1-\lambda\boldsymbol{q}^2)^{1/2}p_i(1-\lambda\boldsymbol{q}^2)^{1/2} + (1-\lambda\boldsymbol{q}^2)^{1/2}p_i(1-\lambda\boldsymbol{q}^2)^{1/2}p_i + p_i(1-\lambda\boldsymbol{q}^2)p_i + (1-\lambda\boldsymbol{q}^2)^{1/2}\boldsymbol{p}^2(1-\lambda\boldsymbol{q}^2)^{1/2}] \\ &+ \tfrac{1}{2}\lambda(\boldsymbol{p}^2\boldsymbol{q}^2 + \boldsymbol{q}^2\boldsymbol{p}^2) - \lambda(\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{p})\}. \end{aligned}$$

Rearrangement of this expression results in equation (29).

where we have replaced the factor $(k_0+\frac{1}{4}\lambda^2\hbar^2)$ by k. With the replacements†

$$p^2 = -\hbar^2\nabla^2 = -\hbar^2\left(\frac{d^2}{dq^2}+\frac{2}{q}\frac{d}{dq}+\frac{L}{q^2}\right) \tag{31}$$

and

$$(\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{p}) = -\hbar^2\left(q^2\frac{d^2}{dq^2}+4q\frac{d}{dq}\right) \tag{32}$$

where L is the usual angular momentum operator

$$L = -\frac{\hbar^2}{\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{\partial^2}{\partial\varphi^2}\right] \tag{33}$$

and assuming that the wavefunction

$$\Psi(q,\theta,\varphi) = \frac{\chi(q)}{q}Y_{lm}(\theta,\varphi), \tag{34}$$

the radial part of the time-independent Schrödinger equation $H\Psi = E\Psi$ becomes

$$(1-\lambda q^2)\frac{d^2\chi}{dq^2}-2\lambda q\frac{d\chi}{dq}+\left(\frac{2(E-\lambda\hbar^2)}{\hbar^2}-\frac{kq^2}{\hbar^2(1-\lambda q^2)}-\frac{l(l+1)}{q^2}\right)\chi = 0. \tag{35}$$

Now with the substitutions

$$\lambda q^2 = t \qquad \text{and} \qquad \chi = (1-t)^{-1/4}R(t) \tag{36}$$

equation (35) reduces to the form

$$t(1-t)\frac{d^2R}{dt^2}+(\tfrac{1}{2}-t)\frac{dR}{dt}+\frac{1}{4}\left[\frac{1}{\lambda}\left(\frac{2E}{\hbar^2}+\frac{9}{4}\lambda+\frac{k}{\lambda\hbar^2}\right)-\left(\frac{k}{\lambda^2\hbar^2}-\frac{1}{4}\right)\frac{1}{1-t}-\frac{l(l+1)}{t}\right]R = 0. \tag{37}$$

One may notice the close connection of this equation to that of the Schrödinger equation of a one-dimensional Pöschl–Teller potential hole (Flügge 1972, p 81). Now with the substitution

$$R(t) = t^{\mu/2}(1-t)^{\nu/2}f(t) \tag{38}$$

equation (37) becomes

$$t(1-t)f''+[\mu+\tfrac{1}{2}-(\mu+\nu+1)t]f'+\tfrac{1}{4}[\rho^2-(\mu+\nu)^2]f = 0. \tag{39}$$

Here

$$\nu = \frac{1}{2}+\frac{\sqrt{k}}{\lambda\hbar} \tag{40a}$$

$$\mu = (l+1) \tag{40b}$$

and

$$\rho^2 = \frac{1}{\lambda}\left(\frac{2E}{\hbar^2}+\frac{9}{4}\lambda+\frac{k}{\lambda^2\hbar^2}\right). \tag{40c}$$

† This corresponds to the normalization $\int \Psi^*\Psi\, d^3q = 1$. If one demands a group invariant normalization $\int \hat{\Psi}^*\hat{\Psi}g^{1/2}\, d^3q = 1$, then one makes the replacement $p_i = -i\hbar\,\partial_i-\frac{1}{4}i\hbar(\ln g)_{,i}$. Then Ψ and $\hat{\Psi}$ are connected by the relation $\Psi = g^{1/4}\hat{\Psi}$. Here $g = \det(g_{ij})$, the determinant of the metric. For details see Charap (1973a).

The general solution to this hypergeometric equation (39) is

$$f = AF(a, b; c; t) + Bt^{1-c}F(a+1-c, b+1-c; 2-c; t) \tag{41}$$

where

$$a = \tfrac{1}{2}(\mu+\nu+\rho) \tag{42a}$$

$$b = \tfrac{1}{2}(\mu+\nu-\rho) \tag{42b}$$

and

$$c = (\nu+\tfrac{1}{2}). \tag{42c}$$

We shall consider the $\lambda > 0$ and $\lambda < 0$ cases separately in what follows.

4.1. *Case (1)*: $\lambda > 0$

In this case the range of q is divided into two regions:

(i) Region I: $0 \leqslant q \leqslant \lambda^{-1/2}$ $(0 \leqslant t \leqslant 1)$

(ii) Region II: $\lambda^{-1/2} \leqslant q \leqslant \infty$ $(1 \leqslant t \leqslant \infty)$.

Applying the boundary conditions to the radial part of the wavefunction in the regions I and II separately (with the aid of equations (34), (35), (40)–(42)), the eigenvalues and eigenfunctions are obtained. We observe that (i) when ρ is real in equation (40*c*) one obtains a discrete set of bound state solutions and (ii) when ρ is imaginary we obtain a continuous spectrum of states representing the scattering solutions.

4.1.1. ρ real. In region I considering the solution (41) we find that the second part of the solution on the right-hand side is always singular at $t = 0$ (because $1-c = -k^{1/2}/\lambda\hbar < 0$). So we choose the constant B to be zero, in which case the remaining part of the solution (41) is well behaved at $t = 0$. However at $t = 1$ this remaining part is singular, in general. To see this we make use of the transformation formula 2.(10-4) of Erdeyli *et al* (1953) which connects the hypergeometric functions with arguments t and $(1-t)$. One finds that under this transformation the solution contains two pieces one of which is singular at $t = 1$. The one and only situation in which this singular piece vanishes is when

$$b = -n_r \qquad \text{or} \qquad \rho^2 = (\mu+\nu+2n_r)^2 \qquad (n_r = 0, 1, 2, \ldots). \tag{43}$$

This is because one of the Γ functions in the denominator of the above-mentioned singular term becomes infinite. Making use of the relations (40) for μ, ν and ρ, we obtain the bound state energy levels to be

$$E_{n_r,l} = (2n_r + l + \tfrac{3}{2})k^{1/2}\hbar + \tfrac{1}{2}\lambda\hbar^2[(2n_r+l)^2 + 3(2n_r+l)]. \tag{44}$$

We may easily show that in the region II no non-trivial solution exists. Thus the correct bound state wavefunction corresponding to the discrete energy levels is

$$\begin{aligned}\Psi(q, \theta, \varphi) &= \frac{1}{q}\chi(q)Y_{lm}(\theta, \varphi)\\ &= A(1-\lambda q^2)^{\frac{1}{2}\nu-\frac{1}{4}}q^{\mu-1}F(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{1}{2}\rho, -n_r; \mu+\tfrac{1}{2}; \lambda q^2)\\ &\quad \times Y_{lm}(\theta, \varphi) \qquad (0 \leqslant q \leqslant \lambda^{-1/2}) \qquad (45a)\\ &= 0 \qquad (q \geqslant \lambda^{-1/2}) \qquad (45b)\end{aligned}$$

where the parameters ν, μ and ρ are as defined in equation (40).

The reduction to the $\lambda \to 0$ spherical harmonic oscillator limit may be performed as below. As the energy level expression (44) shows the correct $\lambda = 0$ limit, we consider only the limiting form of the wavefunction (45). In this limit from (45*a*) we observe that q opens out and fills out the entire range $0 \leqslant q \leqslant \infty$. Now the expression

$$\lim_{\lambda \to 0} (1-\lambda q^2)^{\frac{1}{4}\nu - \frac{1}{4}} = (1-\lambda q^2)^{(\surd k)/2\lambda\hbar} \simeq \exp[-(k^{1/2}/2\hbar)q^2]$$

and further we have the relations $b = -n_r$ and $\alpha = \frac{1}{2}(\mu+\nu+\rho) \simeq k^{1/2}/\lambda\hbar \to \infty$ in the limit $\lambda \to 0$. Also

$$\lambda q^2 = \frac{k^{1/2}}{\hbar} q^2 \frac{1}{\nu} \to 0 \qquad \text{as} \qquad \lambda \to 0.$$

Now using the formula (Erdeyli 1953, p 248) that

$$\lim_{a \to \infty} F(a, b; c; z/a) = \Phi(b, c; z) \tag{46}$$

where Φ is the confluent hypergeometric function in the Humbert symbol, we have

$$\lim_{\lambda \to 0} F(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \tfrac{1}{2}\rho, -n_r; \mu + \tfrac{1}{2}; \lambda q^2) = \Phi(-n_r, l+\tfrac{3}{2}; (k^{1/2}/\hbar)q^2) \tag{47}$$

and equation (45) becomes

$$\lim_{\lambda \to 0} \Psi(q, \theta, \varphi) = A_0 q^l \exp[-(k^{1/2}/2\hbar)q^2]\Phi(-n_r, l+\tfrac{3}{2}; (k^{1/2}/\hbar)q^2 Y_{lm}(\theta, \varphi) \tag{48}$$

where A_0 is the $\lambda \to 0$ limit of the constant A. This is in agreement with the usual spherical oscillator wavefunction (see for example, Flügge 1971, equation (65-12)).

Now the energy levels corresponding to the SU(2) $\otimes$ SU(2) symmetric limit are obtained by putting $k_0 = 0$ in equation (44) as

$$E_{n_r,l}\,(\text{symmetric}) = \tfrac{1}{2}\lambda\hbar^2[(2n_r+l)^2 + 4(2n_r+l) + \tfrac{3}{2}]$$

and the corresponding wavefunctions are given by

$$\Psi(q, \theta, \varphi)|_{k_0=0} = \begin{cases} A_{(k_0=0)}(1-\lambda q^2)^{1/4} q^l F(l+2+n_r, -n_r; l+\tfrac{3}{2}; \lambda q^2) Y_{lm}(\theta, \varphi) & \\ & (0 \leqslant q \leqslant \lambda^{-1/2}) \\ 0 & (q \geqslant \lambda^{-1/2}) \end{cases}$$

4.1.2. ρ imaginary. In this case there exists no regular solution in the region I. However in region II a well behaved solution exists (with $A, B \neq 0$ in equation (41)) and $\rho = \mathrm{i}\sigma$ (σ real). This continuous spectrum of solutions corresponds to energies ranging below the bound state minimum down to $-\infty$. One may also notice that in the corresponding classical situation aperiodic motion (with negative energies) exists with amplitudes exceeding the value $\lambda^{-1/2}$. We also note that this region lies outside the manifold of the SU(2) $\otimes$ SU(2) group.

4.2. Case (2): $\lambda < 0$

In this case the boundary conditions for $\chi(q)$ are to be applied at $q = 0$ and $q = \infty$. Then the analysis proceeds in an analogous manner to the previous case and the bound state levels are given by

$$E_{n_r,l} = (2n_r + l + \tfrac{3}{2})k^{1/2}\hbar - \tfrac{1}{2}|\lambda|[(2n_r+l)^2 + 3(2n_r+l)]\hbar^2 \tag{49a}$$

and the corresponding radial eigenfunctions are given by

$$\chi(q) = A'(1+|\lambda|q^2)^{\frac{1}{2}\nu-\frac{1}{4}}q^{\mu}F(a,b;c;-|\lambda|q^2) \qquad (0 \leqslant q \leqslant \infty). \tag{49b}$$

These states belong to the case $\rho^2 > 0$ in equation (40) and there exists only a finite number of bound states N, where N is the nearest integer to $(\nu+\mu)$.

When ρ is pure imaginary both the constants in (41) may be nonzero. The corresponding solutions represent the scattering states having energies up to $+\infty$ and contain the energy region not covered by the discrete spectrum. Finally, the reduction to the $|\lambda| \to 0$ limit is identical to the previous case.

5. Discussion

We have solved completely the quantum dynamics of a chiral Lagrangian model with non-polynomial (symmetry breaking) interaction wherein all the three degrees of freedom have been included. The bound state energy levels are shown to be of the form

$$E_n = (n+\tfrac{3}{2})k^{1/2}\hbar+\tfrac{1}{2}\lambda\hbar^2(n^2+3n) \qquad (n = 2n_r+l). \tag{50}$$

An interesting question in the context of field theory would be how approximate methods perform in comparison with the exact one. We have already seen that the Bohr–Sommerfeld semiclassical quantization procedure reproduces the form of the bound state levels correctly to within a constant (apart from the fact that k_0 appears instead of k). More interestingly a perturbation theoretic procedure, by taking the three-dimensional harmonic oscillator as the unperturbed system, would also reproduce the exact results at least up to the order λ^2 to which we have carried our calculations for the first few low lying levels. The procedure is to treat the term

$$H_I = \frac{\lambda}{2}\left(-(\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{p})+\frac{k\boldsymbol{q}^4}{1-\lambda\boldsymbol{q}^2}\right) \tag{51}$$

as the perturbed part of the Hamiltonian and then rewrite (51) in terms of the set of operators A_m and $A_m^\dagger$ $(m = -1, 0, 1)$ (see Messiah 1968, p 458) where

$$A_1 = \frac{1}{\sqrt{2}}(a_x-ia_y), \qquad A_0 = a_z \qquad \text{and} \qquad A_{-1} = \frac{1}{\sqrt{2}}(a_x+ia_y). \tag{52}$$

We have verified explicitly that for the first three $(n = 0, 1, 2)$ low lying levels the expression obtained from perturbation theory coincides with the exact one (50) up to order λ^2 (beyond which also one would expect the same thing to happen). The explicit calculations of s wave states $(l = 0)$ to this effect have been given in Mathews and Lakshmanan (1975) and the proof for the other l states is similar, so we refrain from giving the details here.

Another interesting possibility is the comparison of scattering cross sections (as the actual solutions are known in the present case) computed from this model with experimental results. Finally it would be of interest to investigate the full field theoretic case of this model in perturbation theory. It is natural to expect that some of the simplicities of this specific model will also be carried over to the more complicated interacting field case. Investigations are in progress along these lines.

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Appendix

In this appendix we show the equivalence of our quantum Hamiltonian (29) to the one that is obtained from the theory of Lin *et al* (1970). Their theory states that for Lagrangians of the form

$$L = \tfrac{1}{2}\dot{q}_i g_{ij}(q)\dot{q}_j \tag{A.1}$$

the proper quantum mechanical Hamiltonian satisfying the canonical equation of motion would be

$$H = \tfrac{1}{2}\{p_i, \dot{q}_i\} - L - Z(q) \tag{A.2}$$

where

$$Z(q) = -\frac{1}{4}\frac{\partial}{\partial q_i}\left(f_{ij}f_{kl}\frac{\partial g_{kl}}{\partial q_j}\right) - \frac{1}{4}\frac{\partial^2 f_{ij}}{\partial q_i \partial q_j} - \frac{1}{16}f_{ij}f_{kl}f_{lm}\frac{\partial g_{ij}}{\partial q_m}\frac{\partial g_{kl}}{\partial q_m} \tag{A.3}$$

with

$$f_{ij}(q)g_{jk}(q) = \delta_{ik} \qquad (\hbar = 1). \tag{A.4}$$

For our system (1) we have

$$g_{ij}(q) = \delta_{ij} + \frac{\lambda q_i q_j}{1-\lambda \boldsymbol{q}^2} \tag{A.5}$$

so that

$$f_{ij}(q) = \delta_{ij} - \lambda q_i q_j. \tag{A.6}$$

Then it is easy to see that

$$\frac{\partial}{\partial q_i}\left(f_{ij}f_{kl}\frac{\partial g_{kl}}{\partial q_j}\right) = 6\lambda \tag{A.7}$$

$$\frac{\partial^2 f_{ij}}{\partial q_i \partial q_j} = -12\lambda \tag{A.8}$$

and

$$f_{ij}f_{kl}f_{lm}\frac{\partial g_{ij}}{\partial q_m}\frac{\partial g_{kl}}{\partial q_n} = \frac{4\lambda^2 q^2}{1-\lambda q^2}. \tag{A.9}$$

So we have

$$Z(q) = \frac{3}{2}\lambda - \frac{1}{4}\frac{\lambda^2 \boldsymbol{q}^2}{1-\lambda \boldsymbol{q}^2}. \tag{A.10}$$

Then substituting this in (A.2) and using the fact that

$$\dot{q}_i = \tfrac{1}{2}(f_{ij}p_j + p_j f_{ij}) \tag{A.11}$$

we find that

$$H = \frac{1}{2}\left(\boldsymbol{p}^2 - \lambda(\boldsymbol{p}\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\boldsymbol{p}) + \frac{1}{4}\frac{\lambda^2\boldsymbol{q}^2}{1-\lambda\boldsymbol{q}^2}\right) \tag{A.12}$$

and hence the result.

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